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Optimum iterative methods for the solution of singular linear systems arising from the discretization of elliptic P.D.E.'s

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Abstract: This paper refers to three iterative methods, namely the generalized extrapolated Jacobi (GJOR), the generalized successive overrelaxation (GSOR) and the second order 2-cyclic Chebyshev semi-iterative ones, for the solution of a singular linear system $Ax = b$, with $\det(A) = 0$ and b in the range of A. As is known, under certain basic conditions (assumptions), one can determine the various parameters involved in the aforementioned methods so that each one of them semiconverges asymptotically as fast as possible. The theory is applied to the singular linear systems arising from the discretization of the 2-dimensional Neumann and periodic boundary value problems for the Poisson equation in a rectangle. After the verification of the validity of the basic assumptions for each method, the optimum parameters are obtained and by means of them the optimum asymptotic semiconvergence rates for the JOR and SOR methods are determined. These are compared with the convergence rates of the Jacobi and the SOR methods used for the solution of the corresponding nonsingular linear system for the Dirichlet model problem for Poisson equation and various conclusions regarding their relative asymptotic behaviors are drawn.

1. Introduction and preliminaries

For the solution of a linear system

$$Au = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n,n}$, $\det(A) = 0$, u, $b \in \mathbb{C}^n$ and b in the range of A, the splitting

$$A = D - L - U, \tag{1.2}$$

is considered, with D, L, $U \in \mathbb{C}^{n,n}$ any matrices, $\det(D) \neq 0$, and the generalized Jacobi (GJ) iterative scheme associated with (1.2) is constructed (Buoni and Varga [5,6])

$$u^{(m+1)} = Tu^{(m)} + c, \quad m = 0, 1, 2, \ldots, \tag{1.3}$$

where

$$T \equiv D^{-1}(L + U), \quad c \equiv D^{-1}b. \tag{1.4}$$

Let λ_j, $j = 1(1)n$ be the eigenvalues of T and $\sigma(T)$ its spectrum. It is known [3,4] that if the asymptotic semiconvergence factor

$$\gamma(T) \equiv \max_j \left\{ |\lambda_j| : \lambda_j \in \sigma(T) \text{ and } \lambda_j \neq 1 \right\} \tag{1.5}$$

is less than 1 and if $\operatorname{index}(I - T) = 1$ (I the $n \times n$ unit matrix), that is all the elementary divisors of T associated with the eigenvalue 1 are linear, then scheme (1.3) semiconverges. This implies that for any $u^{(0)} \in \mathbb{C}^n$ the iterates $u^{(m)}$ of (1.3) tend to a u which is a solution of

$$u = Tu + c \tag{1.6}$$

and therefore a solution of the original system (1.1).

It was proved in [8] and [9] that under the following assumptions: (i) The point (1, 0) of the complex plane is a vertex of the convex hull H of $\sigma(T)$ (Assumption A) and (ii) index$(I - T) = 1$ (Assumption B), most of the well-known stationary and nonstationary first and second order iterative methods, e.g. the generalized extrapolated Jacobi (GJOR) method, the second order stationary methods of de Pillis' [2,7] and of Manteuffel's type [1,12–14] and the second order Chebyshev semi-iterative methods (noncyclic and cyclic ones) [17,18] etc, can be optimized (and produce, of course, convergent methods) under the same conditions under which the corresponding methods are optimized in the case of nonsingular systems (1.1). Very briefly this optimization is equivalent to the minimization of the asymptotic semiconvergence factor associated with the specific method, which is based on the convex hull $\tilde{H}$ of $\tilde{\sigma}(T) \equiv \sigma(T) \backslash \{1\}$. Also, under the assumptions (iii) $D^{-1}L$ and $D^{-1}U$ in (1.4) are strictly lower and strictly upper triangular matrices (Assumption C) and (iv) T is a weakly 2-cyclic consistently ordered matrix (see Varga [17]) (or equivalently consistently ordered (see Young [18])), the generalized successive overelaxation (GSOR) method etc. can be optimized in exactly the same way the corresponding method for the nonsingular system (1.1) is optimized. This optimization is achieved by basing the analysis for the determination of the optimum overrelaxation parameter on the convex hull $\tilde{H}'$ of $\tilde{\sigma}'(T) \equiv \sigma(T) \backslash \{1, -1\}$. Then one follows either the theory by Kredell [11] for a complex parameter or the algorithm by Young and Eidson [19] (see also [18]) for a real parameter, whichever of them is applicable.

In many practical applications A in (1.1) is an irreducible singular M-matrix so that if one takes in (1.2) $D = \text{diag}(A)$ and L and U the strictly lower and upper triangular parts of A then T is not only a nonnegative matrix but satisfies also Assumptions B (see [3, Theorem 4.16, p. 156]) and C. If, in addition, $\sigma(T)$ is real with one endpoint of (the line segment) H the point (1, 0), Assumption A is satisfied as well so that optimum methods associated with the ones mentioned previously in connection with Assumptions A and B can be applied [8,9]. In such a case the index k of cyclicity of T can only be equal to 1 (T primitive) or to 2 (T weakly 2-cyclic consistently ordered). In the latter case Assumption D will also be satisfied so that the optimum GSOR method, associated with (1.2), can be applied [8]. When T, L and U are defined as was indicated previously, instead of having Generalized methods to deal with we shall be dealing with the classical nongeneralized ones. For example, instead of the GJOR method, we shall have the JOR one etc.

In the next section we consider Poisson equation in a rectangle under Neumann and periodic boundary conditions respectively. We then apply the classical 5-point difference stencil to the nodes of a regular grid of points and determine the eigenvalues of the singular matrix coefficient of the resulting linear system. Having found the corresponding eigenvalue spectra we determine, in the last section, the optimum parameters of the JOR, the SOR (provided it applies) and the 2-cyclic Chebyshev semi-iterative methods for the two problems. Finally various conclusions regarding the asymptotic semiconvergence rates of the JOR and SOR methods, especially when they are compared with the corresponding convergence rates of the analogous Dirichlet problem, are drawn.

2. The Neumann and periodic problems for Poisson equation

Consider the Poisson equation

$$-\left(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2\right) = f(x, y) \tag{2.1}$$

over the rectangle R, with vertices $(0, 0)$, $(l_x, 0)$, (l_x, l_y), $(0, l_y)$, subject to the boundary conditions

$$\partial u/\partial \eta = g(x, y) \tag{2.2a}$$

(Neumann boundary conditions), or to

$$\begin{aligned} u(0, y) &= u(l_x, y), \quad y \in [0, l_y], \\ u(x, 0) &= u(x, l_y), \quad x \in [0, l_x] \end{aligned} \tag{2.2b}$$

(Periodic boundary conditions). In (2.2a), $\partial u/\partial n$ is the outwardly directed first (normal) derivative and the relationship $\int_L g(x, y) \mathrm{d}L = 0$ must be satisfied on the perimeter L of R. For the solution of either problem (2.1)–(2.2a) (Problem I) or (2.1)–(2.2b) (Problem II) we impose a uniform grid on R of mesh sizes h_x and h_y in x- and y-directions respectively so that $M = l_x/h_x$ and $N = l_y/h_y$ are integers. Then we approximate (2.1) at each node $(x_i, y_j) \equiv (ih_x, jh_y)$ by the 5-point difference formula

$$-u_{i-1,j} - u_{i,j-1} + 4u_{i,j} - u_{i,j+1} - u_{i+1,j} = h^2 f(x_i, y_j), \tag{2.3}$$

where for the sake of simplicity it was taken $h_x = h_y = h$. For Problem I, (2.3) is considered for all nodes (i, j), $i = 0(1)M$, $j = 0(1)N$, where to eliminate the values $u_{i,j}$ at points outside R appropriate use of the following formulas, based on (2.2a), is made

$$\begin{aligned} u_{i,-1} - u_{i,1} &= 2hg(ih, 0), & i &= 0(1)M, \\ u_{M+1,j} - u_{M-1,j} &= 2hg(l_x, jh), & j &= 0(1)N, \\ u_{i,N+1} - u_{i,N-1} &= 2hg(ih, l_y), & i &= 0(1)M, \\ u_{-1,j} - u_{1,j} &= 2hg(0, jh), & j &= 0(1)N. \end{aligned} \tag{2.4a}$$

For Problem II, (2.3) is considered for all the nodes (i, j), $i = 0(1)M-1$, $j = 0(1)N-1$ and the values $u_{i,j}$ outside R as well as those on the right and the upper sides of R are eliminated, by virtue of the periodicity of the boundary conditions, by the formulas

$$\begin{aligned} u_{i,-1} &= u_{i,N-1}, & i &= 0(1)M-1, \\ u_{M,j} &= u_{0,j}, & j &= 0(1)N-1, \\ u_{i,N} &= u_{i,0}, & i &= 0(1)M-1, \\ u_{-1,j} &= u_{M-1,j}, & j &= 0(1)N-1. \end{aligned} \tag{2.4b}$$

The study of the two resulting linear systems, where a natural ordering of the nodes concerned is adopted for each problem, will be made in the sequel. It is understood, however, that the common parts, on which the corresponding analyses are based, will be given only once.

2.1. Study of the linear system of Problem I

The totality of $(M+1)(N+1)$ difference equations (2.3)–(2.4a) yields a linear system of the form (1.1) with

$$A = I_L \otimes \tilde{V}_K + \tilde{V}_L \otimes I_K, \tag{2.5}$$

where

$$K = M+1, \quad L = N+1. \tag{2.6}$$

In (2.5) I_J is the $J \times J$ unit matrix, $\tilde{V}_J$ is the $J \times J$ tridiagonal matrix

$$\tilde{V}_J = \begin{bmatrix} 2 & -2 & & & 0 \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ 0 & & & -2 & 2 \end{bmatrix}, \quad J = K, L \tag{2.7}$$

and $\otimes$ denotes tensor product. (For definitions and properties see Halmos [10].) The eigenvalues of $\tilde{V}_J$ in (2.7) are given by the expressions (see Mitchell [15, pp. 39–43])

$$\lambda_j = 4\sin^2(\pi j/(2(J-1))), \quad j = 0(1)J-1. \tag{2.8}$$

By defining now the diagonal matrix

$$C = C_L \otimes C_K, \tag{2.9}$$

with C_J the $J \times J$ diagonal matrix

$$C_J = \begin{bmatrix} \sqrt{2} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \sqrt{2} \end{bmatrix}, \quad J = K, L \tag{2.10}$$

and multiplying (1.1) from the left by C^{-1}, one obtains again a linear system of the same form (1.1), where the new A, u, b are the old $C^{-1}AC$, $C^{-1}u$, $C^{-1}b$ respectively (with the old A and C being defined through (2.5)–(2.7) and (2.9)–(2.10)). Thus for the new matrix A we have

$$A = I_L \otimes V_K + V_L \otimes I_K, \tag{2.11}$$

where V_J is the following $J \times J$ tridiagonal matrix, similar to $\tilde{V}_J$,

$$V_J = C_J^{-1}\tilde{V}_J C_J = \begin{bmatrix} 2 & -\sqrt{2} & & & & & 0 \\ -\sqrt{2} & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & \ddots & & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -\sqrt{2} \\ 0 & & & & & -\sqrt{2} & 2 \end{bmatrix}, \quad J = K, L. \tag{2.12}$$

The components $I_L \otimes V_K$ and $V_L \otimes I_K$ of A in (2.11) possess the following three properties: (i) they are real symmetric matrices; (ii) they commute, since

$$(I_L \otimes V_K)(V_L \otimes I_K) = (I_L V_L) \otimes (V_K I_K) = (V_L I_L) \otimes (I_K V_K) = (V_L \otimes I_K)(I_L \otimes V_K);$$

and, (iii) their eigenvalues, in view of (2.8), (2.11) and (2.12), are given by the expressions

$$\begin{aligned} \lambda_k &= 4\sin^2(\pi k/(2(K-1))), \quad k = 0(1)K-1, \\ \lambda_l &= 4\sin^2(\pi l/(2(L-1))), \quad l = 0(1)L-1, \end{aligned} \tag{2.13}$$

with multiplicities L and K each respectively. Because $I_L \otimes V_K$, $V_L \otimes I_K$ are real symmetric and commute then, according to the theorem of Frobenius, they will possess an orthonormal set of common eigenvectors. If w_k is the eigenvector of V_K of (2.12) corresponding to the eigenvalue λ_k and z_l is the eigenvector of V_L of (2.12) corresponding to the eigenvalue λ_l then the vectors $z_l \otimes w_k$, $l = 0(1)L-1$, $k = 0(1)K-1$ constitute the common set of the eigenvectors of the component of A, in (2.11), as well as of A itself. This is because

$$\begin{aligned} A(z_l \otimes w_k) &= (I_L \otimes V_K + V_L \otimes I_K)(z_l \otimes w_k) = (I_L z_l) \otimes (V_K w_k) + (V_L z_l) \otimes (I_K w_k) \\ &= z_l \otimes (\lambda_k w_k) + (\lambda_l z_l) \otimes w_k = (\lambda_k + \lambda_l)(z_l \otimes w_k). \end{aligned}$$

The equality of the first and the last members in the above series of equalities shows also that the eigenvalues $\lambda_{k,l}$ of A, in (2.11), are given, because of (2.13), by the expressions

$$\begin{aligned} &\lambda_{k,l} = 4\left(\sin^2(\pi k/(2(K-1))) + \sin^2(\pi l/(2(L-1)))\right), \\ &\quad k = 0(1)K-1, \quad l = 0(1)L-1. \end{aligned} \tag{2.14}$$

From the eigenvalues of A one can readily determine the eigenvalues $\mu_{k,l}$ of its Jacobi matrix T given by

$$\mu_{k,l} = \tfrac{1}{2}(\cos(\pi k/(K-1)) + \cos(\pi l/(L-1))), \qquad k = 0(1)K-1, \quad l = 0(1)L-1. \tag{2.15}$$

From the expressions above it is observed that one and only one eigenvalue of T equals 1 (for $k = l = 0$), while all others are real and strictly less than 1. This implies that the Jacobi matrix T of A satisfies both Assumptions A and B of Section 1. (Assumption B can be proved to be valid since A is an irreducible singular M-matrix, as was explained in the last but one paragraph of Section 1.) Therefore for the solution of (1.1) the optimum JOR and 2-cyclic Chebyshev semi-iterative methods can be applied. One can prove, by using simple graph theory (see Varga [17, pp. 186–188]), that T is weakly 2-cyclic consistently ordered so that Assumption D holds. Therefore if one takes as L and U in (1.2) the strictly lower and upper parts of A Assumption C is also satisfied and thus the optimum SOR method is applicable for the solution of the same system.

2.2. *Study of the linear system of Problem II*

The totality of MN difference equations (2.3)–(2.4b) produces a linear system of the form (1.1), with A being given by the general expression (2.11). This time, however,

$$K = M, \qquad L = N \tag{2.16}$$

and V_J is the $J \times J$ circulant matrix defined by

$$V_J = \begin{bmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & 0 & \\ & & \ddots & & \\ & 0 & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{bmatrix}, \quad J = K,\ L, \tag{2.17}$$

with eigenvalues

$$\lambda_j = 4\sin^2(\pi j/J), \quad j = 0(1)J-1 \tag{2.18}$$

(see [16]). The components of A satisfy the same first two properties (i) and (ii) the components of A in (2.11) of Problem I satisfied. However, the eigenvalues of them are given by the following expressions, in view of (2.11), (2.16)–(2.18), different from (2.14),

$$\begin{aligned} \lambda_k &= 4\sin^2(\pi k/K), \quad k = 0(1)K-1, \\ \lambda_l &= 4\sin^2(\pi l/L), \quad l = 0(1)L-1. \end{aligned} \tag{2.19}$$

Following the same analysis as in the previous case of Problem I, one has for the eigenvalues of A that

$$\lambda_{k,l} = 4\left(\sin^2(\pi k/K) + \sin^2(\pi l/L)\right), \qquad k = 0(1)K-1, \quad l = 0(1)L-1. \tag{2.20}$$

From (2.20) one obtains for the eigenvalues $\mu_{k,l}$ of the Jacobi matrix T of A

$$\mu_{k,l} = \tfrac{1}{2}(\cos(2\pi k/K) + \cos(2\pi l/L)), \qquad k = 0(1)K-1, \quad l = 0(1)L-1. \tag{2.21}$$

Again by following a similar reasoning to the one for Problem I it can be proved that T satisfies both Assumptions A and B. Consequently it is concluded that the optimum JOR and 2-cyclic Chebyshev semiiterative methods can be applied for the solution of the corresponding system (1.1) in the present case. For L and U in (1.2) the strictly lower and upper parts of A, Assumption C also holds. Now, if T had neither the lower left and the upper right corner blocks $\frac{1}{4}I_K$ nor the elements $\frac{1}{4}$ at the positions $(1, K)$, $(K, 1)$, $(K+1, 2K)$, $(2K, K+1), \ldots, ((L-1)K+1, LK)$, $(LK, (L-1)K+1)$ it would have exactly the same structure as the matrix T of Problem I and would be a 2-cyclic consistently ordered one. However, the presence of the aforementioned elements may affect the index of cyclicity. In fact, if one constructs a directed graph analogous to the one in Varga [17, p. 187] it will look like the one given in Fig. 1, where only some of its edges have been drawn. As is observed the extra directed arcs $(1, K)$, $(K, 1), \ldots, ((L-1)K+1, LK)$, $(LK, (L-1)K+1)$, $(1, (L-1)K+1)$, $((L-1)K+1, 1), \ldots, (K, LK)$, (LK, K) will retain the 2-cyclic property for the whole graph iff all closed paths formed from nodes like e.g. $1, 2, 3, \ldots, K, 1$ and $1, K+1, 2K+1, \ldots, (L-1)K+1, 1$ have an even number of edges. Evidently these numbers of edges are equal to K and L respectively. It is then implied that if either K or L is odd, T is primitive, so that Assumption D is not satisfied and an optimum semiconvergent SOR method can not be determined. On the other hand, if both K and L are even then T is 2-cyclic consistently ordered, Assumption D holds and an optimum SOR method can be obtained.

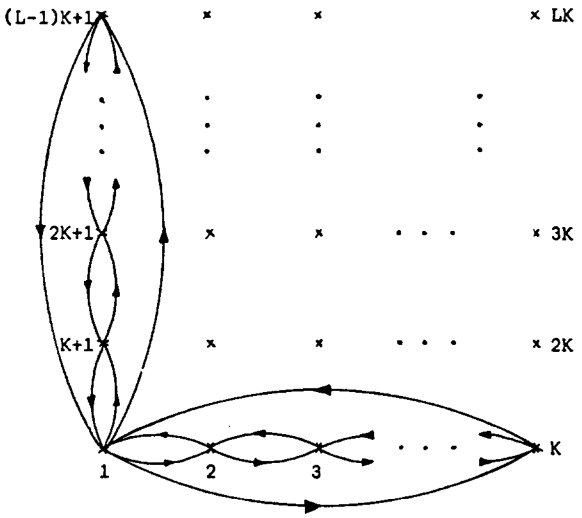

Fig. 1.

3. Optimum semiconvergent methods for Problems I and II

In order to simplify further the expressions for the various parameters, which are to be obtained, it will be assumed that for the rectangle R it is $l_x = l_y = 1$ (and also that $M = N = 1/h$). (It is noted that there is no difficulty to obtain the corresponding expressions in the more general case studied so far.)

3.1. Optimum methods for Problem I

From the expressions (2.15) and (2.9) one can readily find out that all the eigenvalues of T different from 1 satisfy the relationships

$$\alpha \equiv -1 \leqslant \mu \leqslant \tfrac{1}{2}(1+\cos(\pi/N)) \equiv \beta < 1, \tag{3.1}$$

while for all μ different from both 1 and -1 it will be

$$-\beta \equiv -\tfrac{1}{2}(1+\cos(\pi/N)) \leqslant \mu \leqslant \tfrac{1}{2}(1+\cos(\pi/N)) \equiv \beta. \tag{3.2}$$

3.1.1. Optimum JOR method

From the theory in [8] (see also [3]) it is obtained from (3.1) that the optimum extrapolation parameter ω_{opt} and the optimum asymptotic semiconvergence factor are

$$\omega_{\text{opt}} = \frac{2}{2-(\beta+\alpha)} = \frac{4}{5-\cos(\pi/N)}, \qquad \gamma\left(T_{\omega_{\text{opt}}}\right) = \frac{\beta-\alpha}{2-(\beta+\alpha)} = \frac{3+\cos(\pi/N)}{5-\cos(\pi/N)}. \tag{3.3}$$

3.1.2. Optimum SOR method

From the theory in [8] and (3.2) it is for the optimum overrelaxation parameter and the optimum asymptotic semiconvergence factor that

$$\omega_{\text{opt}} = \frac{2}{1+\left(1-\beta^2\right)^{1/2}} = \frac{4}{2+\left(4-\left(1+\cos(\pi/N)\right)^2\right)^{1/2}},$$
$$\gamma\left(Z_{\omega_{\text{opt}}}\right) = \frac{1-\left(1-\beta^2\right)^{1/2}}{1+\left(1-\beta^2\right)^{1/2}} = \frac{2-\left(4-\left(1+\cos(\pi/N)\right)^2\right)^{1/2}}{2+\left(4-\left(1+\cos(\pi/N)\right)^2\right)^{1/2}}. \tag{3.4}$$

3.1.3. Optimum 2-cyclic Chebyshev semi-iterative method

From the theory in [9] and (3.1) the present optimum method is the following

$$\begin{aligned} u^{(2m+1)} &= T_{\omega_{\text{opt}}} u^{(2m)} + \tfrac{1}{4}\omega_{\text{opt}} b, \\ u^{(2m+1)} &= \omega T_{\omega_{\text{opt}}} u^{(2m+1)} + (1-\omega) u^{(2m)} + \tfrac{1}{4}\omega\omega_{\text{opt}} b, \end{aligned} \qquad m = 0, 1, 2, \ldots, \tag{3.5}$$

with

$$T_{\omega_{\text{opt}}} = (1-\omega_{\text{opt}})I + \omega_{\text{opt}} T, \quad \omega = 1/\left(1-\tfrac{1}{2}\gamma^2\right), \tag{3.6}$$

where ω_{opt} and $\gamma \equiv \gamma(T_{\omega_{\text{opt}}})$ are the expressions given in (3.3). The optimum asymptotic semiconvergence factor γ_{opt} of the 2-cyclic scheme (3.5) is

$$\gamma_{\text{opt}} = 1/Q_2(1/\gamma), \tag{3.7}$$

with $Q_2(\cdot)$ the Chebyshev polynomial of degree 2, namely $Q_2(1/\gamma) = 2/\gamma^2 - 1$.

3.2. Optimum methods for Problem II

If we work as in Problem I we can readily determine the optimum parameters when applying the various methods. This time we take into consideration the expressions (2.21) and (2.16). Consequently one has, instead of (3.1),

$$\alpha \equiv -\cos(\pi/N) \leqslant \mu \leqslant \tfrac{1}{2}(1+\cos(2\pi/N)) \equiv \beta < 1 \tag{3.8a}$$

if N is odd, or

$$\alpha \equiv -1 \leqslant \mu \leqslant \tfrac{1}{2}(1+\cos(2\pi/N)) \equiv \beta < 1 \tag{3.8b}$$

if N is even, and instead of (3.2)

$$-\beta \equiv -\tfrac{1}{2}(1+\cos(2\pi/N)) \leqslant \mu \leqslant \tfrac{1}{2}(1+\cos(2\pi/N)) \equiv \beta \tag{3.9}$$

iff N is even.

3.2.1. Optimum JOR method

(i) for N odd it is

$$\omega_{\text{opt}} = \frac{4}{3-\cos(2\pi/N)+2\cos(\pi/N)}, \qquad \gamma\left(T_{\omega_{\text{opt}}}\right) = \frac{1+\cos(2\pi/N)+2\cos(\pi/N)}{3-\cos(2\pi/N)+2\cos(\pi/N)}. \tag{3.10a}$$

(ii) For N even we have

$$\omega_{\text{opt}} = \frac{4}{5-\cos(2\pi/N)}, \qquad \gamma\left(T_{\omega_{\text{opt}}}\right) = \frac{3+\cos(2\pi/N)}{5-\cos(2\pi/N)}. \tag{3.10b}$$

3.2.2. Optimum SOR method (only for N even)

$$\begin{aligned} \omega_{\text{opt}} &= \frac{4}{2+\left(4-(1+\cos(2\pi/N))^2\right)^{1/2}}, \\ \gamma\left(Z_{\omega_{\text{opt}}}\right) &= \frac{2-\left(4-(1+\cos(2\pi/N))^2\right)^{1/2}}{2+\left(4-(1+\cos(2\pi/N))^2\right)^{1/2}}. \end{aligned} \tag{3.11}$$

3.2.3. Optimum 2-cyclic Chebyshev semi-iterative method

The method and the corresponding formulas are given again by (3.5)–(3.7). The only difference now is that we distinguish two cases depending on whether N is odd or even. In the first case the optimum parameters are taken from (3.10a), while in the second one from (3.10b).

Now we are in a position to compare the asymptotic semiconvergence rates of the optimum JOR and SOR methods found previously with the corresponding ones of the Jacobi (J) and SOR methods when the latter methods are applied for the solution of the corresponding nonsingular linear system produced from the Dirichlet model problem, namely the Poisson equation (2.1) over the unit square R with the function $u(x, y)$ known on the boundary of R. (see e.g. Varga [17, pp. 201–204]). (It is noted that the order of the matrix A in the case of Dirichlet problem is $(N-1)^2$, while that of Problem I is $(N+1)^2$ and of Problem II is N^2.) For the linear system in the case of Dirichlet model problem the spectral radii of the J (which is itself the optimum JOR) and optimum SOR iteration matrices are

$$\rho(J) = \cos(\pi/N), \qquad \rho\left(Z_{\omega_b}\right) = (1-\sin(\pi/N))/(1+\sin(\pi/N)), \tag{3.12}$$

so that their asymptotic convergence rates are

$$R_\infty(J) = -\ln \rho(J) \approx \pi^2/(2N^2) \tag{3.13}$$

and

$$R_\infty\left(Z_{\omega_b}\right) = -\ln \rho\left(Z_{\omega_b}\right) \approx 2\pi/N \tag{3.14}$$

(see [17, pp. 203–204]). Using, for $x \to 0$, the formulas

$$\begin{aligned} &\sin x = x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 + O(x^7), \\ &\cos x = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 + O(x^6), \\ &(1-x)/(1+x) = 1 - 2x + 2x^2 + O(x^3), \\ &\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + O(x^4), \\ &(1-x)^{1/2} = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 + O(x^3), \quad \text{etc.,} \end{aligned}$$

which are taken by expanding each function as a Maclaurin series one can obtain. From (3.3)

$$R_\infty\left(T_{\omega_{\text{opt}}}\right) = -\ln \gamma\left(T_{\omega_{\text{opt}}}\right) \approx \pi^2/(4N^2) \tag{3.15}$$

and from (3.4)

$$R_\infty(Z_{\omega_{opt}}) = -\ln\gamma(Z_{\omega_{opt}}) \approx \sqrt{2}\pi/N. \tag{3.16}$$

Results (3.15)–(3.16) give for Problem I that the optimum semiconvergent SOR method is an order of magnitude faster than the optimum semiconvergent JOR method, since

$$R_\infty(Z_{\omega_{opt}})/R_\infty(T_{\omega_{opt}}) \approx (4\sqrt{2}/\pi)N.$$

For Problem II we can find from (3.10a)–(3.10b)

(i) $$R_\infty(T_{\omega_{opt}}) = -\ln\gamma(T_{\omega_{opt}}) \approx \pi^2/N^2, \quad N \text{ odd}, \tag{3.17a}$$

(ii) $$R_\infty(T_{\omega_{opt}}) = -\ln\gamma(T_{\omega_{opt}}) \approx \pi^2/N^2, \quad N \text{ even}, \tag{3.17b}$$

and from (3.11)

$$R_\infty(Z_{\omega_{opt}}) = -\ln\gamma(Z_{\omega_{opt}}) \approx 2\sqrt{2}\pi/N \quad \text{for } N \text{ even only}. \tag{3.18}$$

In case N is even it is from (3.17b) and (3.18)

$$R_\infty(Z_{\omega_{opt}})/R_\infty(T_{\omega_{opt}}) \approx (2\sqrt{2}/\pi)N,$$

so that the optimum semiconvergent SOR method is an order of magnitude faster than the corresponding JOR one.

The results concerning the asymptotic (semi)convergence rates R_∞, as $N \to \infty$, for the various optimum JOR and SOR methods examined so far are illustrated in Table 1. Using the results from Table 1 it is concluded that the optimum semiconvergent JOR method for the Neumann problem is half ($\frac{1}{2}$) as fast as the convergent method for the Dirichlet problem since

$$R_\infty(T_{\omega_{opt}})/R_\infty(J) \approx \tfrac{1}{2},$$

while for the SOR method the corresponding factor is $\frac{1}{2}\sqrt{2}$, because

$$R_\infty(Z_{\omega_{opt}})/R_\infty(Z_{\omega_b}) \approx \tfrac{1}{2}\sqrt{2}.$$

Comparing now the periodic and Dirichlet problems using again the results illustrated in Table 1 we obtain

$$R_\infty(T_{\omega_{opt}})/R_\infty(J) \approx 2(!),$$

Table 1
Order of the asymptotic (semi)convergence rates R_∞.

Problem	Order of linear system	Iterative method	Type of convergence	R_∞
Neumann	$(N+1)^2$	Optimum JOR	Semiconvergence	$\pi^2/(4N^2)$
		Optimum SOR	Semiconvergence	$\sqrt{2}\pi/N$
Periodic	N^2	Optimum JOR	Semiconvergence	π^2/N^2
		Optimum SOR (N even)	Semiconvergence	$2\sqrt{2}\pi/N$
Dirichlet	$(N-1)^2$	J	Convergence	$\pi^2/(2N^2)$
		Optimum SOR	Convergence	$2\pi/N$

so that the optimum semiconvergent JOR method for Problem II is two times faster than the J method for the corresponding Dirichlet problem. Also it is, for N even,

$$R_\infty(Z_{\omega_{opt}})/R_\infty(Z_{\omega_b}) \approx \sqrt{2}\,(!).$$

Consequently the optimum semiconvergent SOR method for Problem II is $\sqrt{2}$ times faster than the optimum SOR method for the Dirichlet problem.

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